
Submit the solutions in groups of two at the lecture on Tuesday, 2018-06-05

Exercise 1. Let $1 < p < \infty$. Let $\varphi \in \mathcal{S}(\mathbb{R}^d)$ be a Schwartz function with $\int \varphi = 0$ and let $\varphi_t(x) = t^{-d}\varphi(t^{-1}x)$.

(a) Using the estimate for the intrinsic square function from the lecture as a black box prove that for all $f \in L^p(\mathbb{R}^d)$ we have

$$\Big\| \big(\sum_{j\in\mathbb{Z}} |\phi_{2^j} * f|^2 \big)^{1/2} \Big\|_p \lesssim \|f\|_p, \tag{1}$$

where the implicit constant is independent of f. Hint: decompose φ into pieces supported on annuli.

(b) Using (1) as a black box deduce also the continuous version

$$\Big\| \big(\int_0^\infty |\varphi_t * f|^2 \frac{\mathrm{d}t}{t} \big)^{1/2} \Big\|_p \lesssim \|f\|_p.$$

Exercise 2. Let $1 < p < \infty$. Let $\psi \in \mathcal{S}(\mathbb{R}^d)$ be a Schwartz function such that $\widehat{\psi}$ is supported on the annulus $\{2^{j-1} \le |\xi| \le 2^{j+1}\}$ and

$$\sum_{j\in\mathbb{Z}} \widehat{\psi}(2^j \xi) = 1 \text{ for every } \xi \in \mathbb{R}^d \setminus \{0\}.$$

(a) Show that for every $f \in L^p(\mathbb{R}^d)$ we have

$$f = \sum_{j\in\mathbb{Z}} \psi_{2^j} * f = \lim_{J\to\infty} \sum_{|j|\le J} \psi_{2^j} * f \tag{2}$$

with convergence in $L^p(\mathbb{R}^d)$.

(b) Show that

$$\|f\|_p \lesssim \Big\| \big(\sum_{j\in\mathbb{Z}} |\psi_{2^j} * f|^2 \big)^{1/2} \Big\|_p$$

for every $f \in L^p(\mathbb{R}^d)$ with an implicit constant that does not depend on f. Hint: use duality and Exercise 1a with an appropriately chosen function ϕ.

(c) Show that the series (2) converges unconditionally for every $f \in L^p(\mathbb{R}^d)$.