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Submit the solutions in groups of two at the lecture on Tuesday, 2018-07-03

Exercise 1. Show that the $L^p \to L^q$ results for the averaging operator

$$Af(y) = \int_{\partial B(y,1)} f(x)\, d\sigma(x)$$

can only hold when $(1/p, 1/q) \in T$ where T is the closed triangle with vertices $(0,0)$, $(1,1)$ and $(\frac{d}{d+1}, \frac{1}{d+1})$. Do this in $\mathbb{R}^3$ following the instructions given below.

(a) Suppose $f(x)$ vanishes for small x and $f(x) > |x|^{-r}$, for $|x| > 1$. Use this function to verify that $p \leq q$ must hold (the side of the triangle joining $(0,0)$ and $(1,1)$).

(b) Next let $f = 1_{B(0,\delta)}$. Verify $\|f\|_{L^p} \approx \delta^{3/p}$ and $\|Af\|_{L^q} \gtrsim \delta^{2+1/q}$. Use this to conclude one of the restrictions on $(1/p, 1/q)$ coming from the lower sides of the triangle.

(c) Use duality and (b) to finish the proof.

Exercise 2. Let a compact hypersurface $M = \{x \in B : \rho(x) = 0\}$ with $\rho \in C^\infty(\mathbb{R}^d)$ and some ball $B \subset \mathbb{R}^d$ contain a neighborhood in one $d-1$-hyperplane (for example $\{x_d = 1\}$). Show that in this case $|\widehat{\sigma}(\xi)| \lesssim |\xi|^{-\epsilon}$ cannot hold for any $\epsilon > 0$. Here σ is the surface measure of M.